Radiation torque on a birefringent sphere caused by an electromagnetic wave

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We present an exact ab initio calculation of the optical torque on a spherical uniaxially birefringent particle of arbitrary size illuminated by plane electromagnetic wave of arbitrary polarization mode and direction of propagation. The calculation is based on the extended Mie theory and the Maxwell stress tensor formalism. The expression for evaluating radiation torque is derived for arbitrary (absorbing and lossless) isotropic surrounding medium. The dependence of the optical torque on the incident angle, the polarization mode, the material birefringence, as well as the particle size, has been systematically investigated. For normal illumination, namely, with the incident wave vector \( \mathbf{k}_0 \) perpendicular to the extraordinary axis (EA) of the particle, the optical torque \( \Gamma \) caused by a linearly polarized (LP) incident wave always shows the angle dependence \( \Gamma = \Gamma_0 \sin 2\varphi_e \). Here, \( \varphi_e \) is the angle between the EA and the incident electric field, whereas \( \Gamma_0 \) may take positive or negative values, dependent on \( n_o, n_e \), and the particle size. In the small particle limit, \( \Gamma \) versus particle radius \( a \) displays different power law behaviors, \( \Gamma \sim a^3 \) and \( \Gamma \sim a^6 \), for LP and circularly polarized (CP) incident waves, respectively, while for small material birefringence \( |\Delta n| = |n_o - n_e| \), linear and square laws, \( \Gamma \sim |\Delta n| \) and \( \Gamma \sim |\Delta n|^2 \), are found for the LP and the CP incident modes, respectively.

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I. INTRODUCTION

It is well known that electromagnetic wave transports angular momentum as well as linear momentum. When scattered, in addition to optical force caused by the transfer of linear momentum, the electromagnetic wave also exerts on the scatterer a torque due to the transfer of angular momentum. In a certain case, this torque may make the scatterer rotate \([1]\), which, when combined with optical tweezers \([2]\), introduces a unique and simple handle to control both the location and the orientation of a particular microsized particle. The advance has led to a wide variety of applications, including the possibility of rotating biological structure and developing optically driven and controlled micromachines. It has been shown \([3]\) that the optical torque would vanish for an isotropic nonabsorbing spherical scatterer. So, early rotational micromanipulation was achieved by using absorbing particles \([4,5]\). To overcome the heating problem in the absorbing particle that limits the achievable rotation rate, optical systems have recently been designed with lossless geometrically anisotropic \([6–11]\) and optically anisotropic (birefringent) particles \([12–19]\). In particular, optical apparatuses have been demonstrated \([18,19]\) that can apply and accurately measure the torque exerted by laser beam on a birefringent particle.

Although intensively investigated experimentally, the optical torque on a birefringent particle was theoretically evaluated mostly in two limiting cases: \( \lambda \ll a \) \([12,20]\) and \( \lambda \gg a \) \([18]\). Here, \( \lambda \) denotes the incident wavelength, and \( a \) the dimension of the scattering particle, e.g., the radius of spherical scatterer. Friese and co-workers \([12]\) proposed that the optical torque per unit area is given by

\[
\Gamma = -\frac{\epsilon c}{2\omega} E_0^2 \sin[k_0 d(n_o - n_e)] \cos 2\phi \sin 2\varphi_e
+ \frac{\epsilon c}{2\omega} E_0^2 [1 - \cos[k_0 d(n_o - n_e)]] \sin 2\phi,
\]

where \( \epsilon \) is the electric permittivity, \( k_0 \) the free-space wave number, \( \omega \) the angular frequency of incident wave, \( c \) the light velocity, \( d \) the thickness of the particle, and \( n_o \) and \( n_e \) are the refractive indices of the birefringent particle in, respectively, the ordinary and extraordinary directions, whereas \( \varphi_e \) is the angle between the extraordinary axis (EA) of the birefringent particle and the incident electric field, and finally, \( \phi \) describes the degree of ellipticity of the incident light \([12]\), with \( \phi=0 \) or \( \pi/2 \) corresponding to linearly polarized (LP) and \( \phi=\pi/4 \) circularly polarized (CP) light. The expression, ideally suited for a flat disk, is based on ray optics, which is valid in the limiting case \( \lambda \ll a \). La Porta and Wang presented another expression \([18]\), which was implicit in the earlier analysis of more elementary cases (see, e.g., Ref. \([3]\))

\[
\Gamma = \hat{q} \Gamma_0 \sin 2\varphi_e,
\]

where \( \hat{q} \) is a unit vector normal to the electric field and the polarization induced on the scatterer, and \( \Gamma_0 \) is the maximum magnitude of the optical torque. Equation (2) was derived in the limiting case \( \lambda \gg a \). The two expressions meet difficulty for the intermediate case, where the size of the scattering particle \( a \) and the incident wavelength \( \lambda \) are of the same order. It is therefore desirable to have a complete solution to the optical torque problem on birefringent particle.

The purpose of this paper is to present an exact ab initio calculation of the radiation torque on a spherical uniaxially birefringent particle of arbitrary size, caused by an incident
plane electromagnetic wave with arbitrary polarization mode and direction of propagation. The dependence of the radiation torque on the incident angle, incident polarization mode, incident wavelength, as well as the material birefringence has been systematically investigated. The results verify the wave-plate behavior in the transfer of angular momentum from light to the scatterer. It is found that the angle dependence of Eq. (2), which is derived in the limiting case $\lambda \gg a$, remains valid for arbitrary particle size when the particle is subject to normal illumination, namely, with the incident wave vector $k_0$ normal to the EA, by the LP incident wave. However, the maximum magnitude of the optical torque may change sign as particle size increases, in qualitative agreement with the first term of Eq. (1) derived based on ray optics. Due to the shape effect, at some particular ranges of particle size, the EA can be aligned by the LP incidence, namely, with the incident angle, incident polarization mode, and direction of propagation. The dependence of the radiation torque on the incident angle, incident polarization mode, and direction of propagation. The dependence of the radiation torque on the incident angle, incident polarization mode, and direction of propagation. The dependence of the radiation torque on the incident angle, incident polarization mode, and direction of propagation.

The constitutive relations between the electric displacement vector $D_i$ and the magnetic field $H_j$ inside the particle are given, for a birefringent particle, by

$$D_i = \varepsilon \cdot E_i, \quad B_j = \mu \cdot H_j,$$

where the permittivity tensor $\varepsilon$ is given by

$$\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + u \end{pmatrix},$$

and $\mu(\mu_s)$ is the scalar permittivity (permeability). For simplicity, $\mu_s$ is set to be $\mu_0$, the permeability of the surrounding medium, while $\varepsilon = \varepsilon_\text{a}$, $\varepsilon_\text{p}$ denoting the permittivity outside particle. It follows from (3) and (4) that $D_i$ satisfies the wave equation

$$\nabla \times \nabla \times (\varepsilon^{-1} \cdot D_i) - k_s^2 D_i = 0,$$

with $k_s^2 = \omega^2 \varepsilon \mu_s$ and

$$\varepsilon^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + u \end{pmatrix}.$$  

The divergenceless property (3c) suggests that $D_i$ be expanded in terms of the vector spherical wave functions (VSWFs) $M^{(1)}_{mn}(k, r)$ and $N^{(1)}_{mn}(k, r)$ [21–23]

$$D_i = \sum_{n,m} E_{mn} [c_{mn} M^{(1)}_{mn}(k, r) + d_{mn} N^{(1)}_{mn}(k, r)],$$

where $c_{mn}$ and $d_{mn}$ are the expansion coefficients, and $k$ is as yet undetermined. In general, there are three kinds of VSWFs: $M^{(0)}_{mn}(k, r)$, $N^{(0)}_{mn}(k, r)$, and $L^{(0)}_{mn}(k, r)$. The divergenceless property of $D_i$ implies that it does not involve $L_{mn}$, thereby simplifying the algebra involved. The three kinds of VSWFs are defined for $J=1$ and 3 as in Refs. [21–23]. Except otherwise explicitly specified, hereinafter the summation $\sum_{n,m}$ implies that $n$ runs from 1 to $+\infty$ and $m$ from $-n$ to $+n$ for each $n$. The implication of $\varepsilon_{\mu_s}$ is similar. The prefactor $E_{mn}$ is given by $E_{mn} = i^{m+n} \cdot C_{mn}$, with $E_{0}$ characterizing the amplitude of the electric field of incident wave and [22]

$$C_{mn} = \left[ \frac{2m + 1}{\pi n!} \right]^{1/2} \cdot \frac{(n-m)!}{(n+m)!}.$$  

By using the properties of VSWFs, it can be worked out that [21]

$$\varepsilon^{-1} \cdot M^{(0)}_{mn} = \sum_{\mu=0}^{+\mu} \sum_{\nu=0}^{+\nu} \left[ \varepsilon^{(m)}_{\mu \nu} M_{\mu \nu} + \varepsilon^{(m)}_{\mu \nu} N_{\mu \nu} + f^{(m)}_{\mu \nu} L_{\mu \nu} \right],$$

$$\varepsilon^{-1} \cdot N^{(0)}_{mn} = \sum_{\mu=0}^{+\mu} \sum_{\nu=0}^{+\nu} \left[ \varepsilon^{(m)}_{\mu \nu} M_{\mu \nu} + \varepsilon^{(m)}_{\mu \nu} N_{\mu \nu} + f^{(m)}_{\mu \nu} L_{\mu \nu} \right].$$

Therefore, one has
\[ \mathbf{D}_f = \sum l \alpha_l \mathbf{V}_l, \]  

where the expansion coefficients \( \alpha_l \) are to be determined by matching the boundary conditions at the surface of the sphere. With \( \mathbf{D}_f \) given by (19), it follows from (3a) and (4) that \( \mathbf{E}_f \) and \( \mathbf{H}_f \) fields can be written as

\[ \mathbf{E}_f = \frac{1}{\varepsilon_\infty} \mathbf{D}_f = \sum_{n,m} i E_{mn} \sum l \alpha_l \begin{bmatrix} c_{mn,l} \mathbf{M}^{(1)}_{mn}(k, r) + d_{mn,l} \mathbf{N}^{(1)}_{mn}(k, r) \end{bmatrix} \times \mathbf{N}_{mn}^{(1)}(k, r) + \frac{w_{mn,l}}{\lambda_l} \mathbf{L}_{00}^{(1)}(k, r) \] 

\[ + \sum l i \alpha_l \begin{bmatrix} w_{00,l} \mathbf{L}_{00}^{(1)}(k, r) \end{bmatrix}, \]

\[ \mathbf{H}_f = -\frac{i}{\omega \mu_0} \nabla \times \mathbf{E}_f = \frac{1}{\omega \mu_0} \sum_{n,m} \sum l k_l c_{mn,l} d_{mn,l} \mathbf{M}^{(1)}_{mn}(k, r) + c_{mn,l} \mathbf{N}^{(1)}_{mn}(k, r), \]

with

\[ w_{mn,l} = \sum_{v,\mu} E_{vm} \begin{bmatrix} \mu \nu \mathbf{M}^{(1)}_{mn}(k, r) + \mu \nu \mathbf{N}^{(1)}_{mn}(k, r) \end{bmatrix}, \]

\[ w_{00,l} = \sum_{v,\mu} E_{vm} \begin{bmatrix} \mu \nu \mathbf{L}_{00}^{(1)}(k, r) + \mu \nu \mathbf{L}_{00}^{(1)}(k, r) \end{bmatrix} = \frac{-1}{2 \mu_0} d_{00,l}. \]

Notice that, since \( \mathbf{E}_f \) no longer satisfies \( \nabla \cdot \mathbf{E}_f = 0 \), its expansion includes \( \mathbf{L}_{00} \) terms that are absent in the isotropic case.

The scattered fields \( \mathbf{E}_s, \mathbf{H}_s \), and incident fields \( \mathbf{E}_{inc}, \mathbf{H}_{inc} \) in the isotropic surrounding medium have the same form as in the Mie solution [23,24]. However, since the form of the permittivity tensor adopted [see (5)] implies that the EA of birefringent particle is along the z axis, for arbitrary incident direction of plane wave, the expansions of the fields are not limited to \( m = \pm 1 \) modes.

In terms of VSWFs, the scattered fields \( \mathbf{E}_s, \mathbf{H}_s \) are expanded as

\[ \mathbf{E}_s = \sum_{n,m} i E_{mn} \begin{bmatrix} a_{mn} \mathbf{N}^{(3)}_{mn}(k_0, r) + b_{mn} \mathbf{M}^{(3)}_{mn}(k_0, r) \end{bmatrix}, \]

\[ \mathbf{H}_s = \frac{k_0}{\omega \mu_0} \sum_{n,m} E_{mn} \begin{bmatrix} b_{mn} \mathbf{N}^{(3)}_{mn}(k_0, r) + a_{mn} \mathbf{M}^{(3)}_{mn}(k_0, r) \end{bmatrix}, \]

where \( k_0^2 = \omega^2 \varepsilon_0 \mu_0 \) with \( \varepsilon_0 \) and \( \mu_0 \) being, respectively, the scalar permittivity and permeability of the surrounding medium. The expansion coefficients \( a_{mn} \) and \( b_{mn} \) are to be determined by matching boundary conditions.

Suppose that the particle is illuminated by a plane wave characterized by the incident wave vector \( \mathbf{k}_0 \), with

\[ \mathbf{k}_0 = k_0 (\sin \theta_s \cos \phi_s \mathbf{e}_x + \sin \theta_s \sin \phi_s \mathbf{e}_y + \cos \theta_s \mathbf{e}_z), \]

where \( \mathbf{e}_x, \mathbf{e}_y, \) and \( \mathbf{e}_z \) are three unit base vectors of the Cartesian coordinate system and \( \theta_s, \phi_s \) is the polar (azimuthal) angle of \( \mathbf{k}_0 \), as schematically shown in Fig. 1. The electric and magnetic fields of the incident plane wave are then
\[ E_{\text{inc}} = E_0 (p_{\phi} \hat{\theta}_k + p_{\phi} \hat{\phi}_k) e^{i k_0 z}, \]
\[ H_{\text{inc}} = \frac{k_0}{\omega \mu_0} E_0 (p_{\phi} \hat{\phi}_k - p_{\phi} \hat{\theta}_k) e^{i k_0 z}, \]

where \( \hat{p} = (p_{\phi} \hat{\theta}_k + p_{\phi} \hat{\phi}_k) \) is the normalized complex polarization vector, with \( |\hat{p}| = 1 \), and the unit vectors \( \hat{\theta}_k \) and \( \hat{\phi}_k \) are defined in the direction of increasing \( \theta_k \) and \( \phi_k \) such as to constitute a right-hand base system together with \( \hat{k}_0 = k_0 / k_0 \), as shown in Fig. 1, namely

\[ \hat{k}_0 \times \hat{\theta}_k = \hat{\phi}_k, \quad \hat{\theta}_k \times \hat{\phi}_k = \hat{k}_0, \quad \hat{\phi}_k \times \hat{k}_0 = \hat{\theta}_k. \]

In terms of VSWFs, the incident fields (\( E_{\text{inc}}, H_{\text{inc}} \)) read

\[ E_{\text{inc}} = -\sum_{n,m} i E_{mn} [p_{mn} N_m^{(1)}(k_0, r) + q_{mn} M_m^{(1)}(k_0, r)], \]
\[ H_{\text{inc}} = -\frac{k_0}{\omega \mu_0} \sum_{n,m} E_{mn} [q_{mn} N_m^{(1)}(k_0, r) + p_{mn} M_m^{(1)}(k_0, r)]. \]

The expansion coefficients \( p_{mn} \) and \( q_{mn} \) are

\[ p_{mn} = [p_{\theta} \bar{\sigma}_{mn}(\cos \theta_k) - ip_{\phi} \sigma_{mn}(\cos \theta_k)] e^{-im\phi_k}, \]
\[ q_{mn} = [p_{\theta} \bar{\sigma}_{mn}(\cos \theta_k) - ip_{\phi} \sigma_{mn}(\cos \theta_k)] e^{-im\phi_k}, \]

where the regular angular functions \( \bar{\sigma}_{mn}(\cos \theta) \) and \( \sigma_{mn}(\cos \theta) \) are defined based on the first kind of associated Legendre functions \( P_{n}^{(m)}(\cos \theta) \) [22-24]

\[ \bar{\sigma}_{mn}(\cos \theta) = \sigma_{mn}(\cos \theta) = C_{mn} \frac{m}{\sin \theta} P_{n}^{(m)}(\cos \theta), \]

\[ \bar{\pi}_{mn}(\cos \theta) = \sigma_{mn}(\cos \theta) = C_{mn} \frac{d}{d\theta} P_{n}^{(m)}(\cos \theta), \]

with \( C_{mn} \) defined in (9).

Matching boundary conditions at the surface of sphere, and after some algebra, one arrives at the equations that serve to determine the expansion coefficients \( \alpha_i, \mu_{mn}, \) and \( b_{mn} \), based on \( p_{mn} \) and \( q_{mn} \)

\[ \frac{1}{m_\sigma} \sum_{l} \frac{1}{k_l} j_{l_1}(k_l m_\sigma, \eta) w_{mn, l_\sigma} \alpha_l + \xi_{\alpha}(\eta) a_{mn} \]
\[ + \frac{1}{m_\sigma} \sum_{l} \frac{1}{k_l} j_{l_1}(k_l m_\sigma, \eta) d_{mn, l_\sigma} \alpha_l = \psi_{\alpha}(\eta) p_{mn}, \]
\[ \xi_{\alpha}(\eta) b_{mn} + \frac{1}{m_\sigma} \sum_{l} \frac{1}{k_l} j_{l_1}(k_l m_\sigma, \eta) c_{mn, l_\sigma} \alpha_l = \psi_{\alpha}(\eta) q_{mn}, \]
\[ \frac{1}{m_\sigma} \sum_{l} \frac{1}{k_l} j_{l_1}(k_l m_\sigma, \eta) d_{mn, l_\sigma} \alpha_l = \psi_{\alpha}(\eta) p_{mn}, \]
\[ \frac{1}{m_\sigma} \sum_{l} \frac{1}{k_l} j_{l_1}(k_l m_\sigma, \eta) c_{mn, l_\sigma} \alpha_l = \psi_{\alpha}(\eta) q_{mn}, \]

where

\[ \eta = k_0 \alpha, \quad m_\sigma = \sqrt{1 + u_\sigma k_0}, \quad k_l = k_0, \quad \lambda_l = \frac{k_0^2}{k_l} = \frac{1}{k_l}, \]

with \( \alpha \) the radius of spherical particle. The Ricatti-Bessel functions \( \psi_{\alpha}(z), \xi_{\alpha}(z), \chi_{\alpha}(z) \) are given by [24]

\[ \psi_{\alpha}(z) = z j_{\alpha}(z), \quad \xi_{\alpha}(z) = z k_{\alpha}^{(1)}(z), \quad \chi_{\alpha}(z) = -z y_{\alpha}(z). \]

Some details about numerical solution of (28) can be found in Ref. [21].

### B. Radiation torque

With the incident fields and the scattered field given by (25) and (21), respectively, the total external field outside the particle reads

\[ \mathbf{E}_e = \sum_{n,m} i E_{mn} [a_{mn} N_m^{(1)}(k_0, r) + b_{mn} M_m^{(1)}(k_0, r) - p_{mn}] \]
\[ \times N_m^{(1)}(k_0, r) - q_{mn} M_m^{(1)}(k_0, r)], \]
\[ \mathbf{H}_e = \frac{k_0}{\omega \mu_0} \sum_{n,m} E_{mn} [b_{mn} N_m^{(1)}(k_0, r) + a_{mn} M_m^{(1)}(k_0, r) - q_{mn}] \]
\[ \times N_m^{(1)}(k_0, r) - p_{mn} M_m^{(1)}(k_0, r)], \]

which is explicitly reduced to

\[ \mathbf{E}_e = i \frac{k_0}{\kappa_\rho} \sum_{n,m} E_{mn} e^{im\phi} \left[ \frac{n(n + 1)}{k_0^2} \mathbf{U}_m^{(p)} e_\rho + (\mathbf{U}_m^{(p)} \mathbf{U}_m^{(p)}) e_\phi + \mathbf{V}_m^{(p)} (\mathbf{U}_m^{(p)} \mathbf{U}_m^{(p)}) e_\phi \right], \]

FIG. 1. Geometry of the scattering problem.
\[ H_x = \frac{k_0}{\omega \mu_0} \sum_{n,m} \sum_{\nu,\mu} E_{mn} e^{i \nu \phi} \left[ \frac{n(n+1)}{k_0} \gamma_{mn}^\nu P_{\nu}^\nu e_r + \left( \gamma_{mn}^\nu + \nu \tau_{mn}^\nu \right) e_r \right] + \mathcal{U}_{mn} \tau_{mn}^\nu e_\theta + \left( \gamma_{mn}^\nu + \nu \tau_{mn}^\nu - \mathcal{U}_{mn} \tau_{mn}^\nu \right) e_\phi, \]

where

\[ \mathcal{U}_{mn} = a_{mn} \xi_{mn}(k_0^r) - p_{mn} \eta_{mn}(k_0^r), \]

\[ \mathcal{U}_{mn}^* = a_{mn}^* \xi_{mn}(k_0^r) - p_{mn}^* \eta_{mn}(k_0^r), \]

\[ \gamma_{mn} = b_{mn} \xi_{mn}(k_0^r) + q_{mn} \eta_{mn}(k_0^r), \]

\[ \gamma_{mn}^* = b_{mn}^* \xi_{mn}(k_0^r) + q_{mn}^* \eta_{mn}(k_0^r). \]

The time-averaged Maxwell stress tensor is then, for isotropic medium

\[ \hat{T} = \frac{1}{2} \text{Re} \left[ E_x D_x^* + H_x B_x^* - \frac{1}{2} (E_x \cdot D_x^* + H_x \cdot B_x^*) \hat{I} \right], \]

where the superscript $*$ denotes the complex conjugate, and $\hat{I}$ is the unit dyadic. The time-averaged torque $\Gamma$ on the spherical scatterer can be evaluated by [25,26]

\[ \Gamma = - \int \int dS \oint \hat{k} \cdot d\hat{S} = - \int \int [e_r \cdot \hat{k}] d\Omega, \]

where $e_r = r/r$ is the unit vector in radial direction with $r = |r|$, and the time-averaged angular momentum flux tensor $\hat{K}$ reads [26]

\[ \hat{K} = \hat{T} \cdot (r \times \hat{I}) = \hat{T} \times r. \]

As a result, the time-averaged torque becomes

\[ \Gamma = - \int \int [e_r \cdot (\hat{T} \times r)] d\Omega = - \int \int [r \times (\hat{T} \cdot e_r)] d\Omega. \]

After some algebra, the Cartesian components of the torque can be worked out as

\[ \Gamma = \text{Re} \left[ \Gamma_{x}^{(1)} + \Gamma_{x}^{(2)} \right] e_x + \left[ \Gamma_{y}^{(1)} + \Gamma_{y}^{(2)} \right] e_y + \left[ \Gamma_{z}^{(1)} + \Gamma_{z}^{(2)} \right] e_z, \]

where

\[ \Gamma_{x}^{(1)} = \frac{1}{2} r^3 \sum_{n,m} \sum_{\nu,\mu} \left[ \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(1)} - \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(1)} \right], \]

\[ \Gamma_{y}^{(1)} = \frac{1}{2} r^3 \sum_{n,m} \sum_{\nu,\mu} \left[ \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(2)} - \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(2)} \right], \]

\[ \Gamma_{z}^{(1)} = \frac{1}{2} r^3 \sum_{n,m} \sum_{\nu,\mu} \left[ \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(3)} - \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(3)} \right], \]

\[ \Gamma_{x}^{(2)} = \frac{1}{2} r^3 \sum_{n,m} \sum_{\nu,\mu} \left[ \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(4)} - \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(4)} \right], \]

\[ \Gamma_{y}^{(2)} = \frac{1}{2} r^3 \sum_{n,m} \sum_{\nu,\mu} \left[ \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(5)} - \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(5)} \right], \]

\[ \Gamma_{z}^{(2)} = \frac{1}{2} r^3 \sum_{n,m} \sum_{\nu,\mu} \left[ \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(6)} - \mathcal{U}_{mn} \mathcal{U}_{mn}^* \mathcal{W}_{\nu \mu}^{(6)} \right], \]

where $\mathcal{W}_{\nu \mu}^{(1,2,3,4,5,6)}$ are integrals involving two regular angular functions $\tau_{mn}(\cos \theta)$ and $\tau_{mn}(\cos \phi)$ given in the Appendix.

It should be noted that the integrations in Eqs. (35) and (37) are over the outer surface of the spherical particle, so $\mathcal{U}_{mn}$, $\mathcal{U}_{mn}^*$, $\mathcal{U}_{mn}$, and $\gamma_{mn}$ given by Eq. (33), as well as $\Gamma_{x,y,z}^{(1,2)}$ given in (39) and (40), should be calculated at the sphere surface $\mathcal{r} = a$. If so evaluated, the formulas for radiation torque Eqs. (38)–(40) hold for both absorbing and lossless surrounding medium. They are presented here for arbitrary isotropic surrounding medium. If the background medium is lossless, with both permittivity and permeability being real numbers, then the integration (35) can be performed at spherical surface with arbitrary radius $\mathcal{r} > a$, due to conservation of momentum and angular momentum. As a result, the integration is usually evaluated in the limit $\mathcal{r} \rightarrow \infty$ for lossless surrounding medium, where the field expressions become much simpler by using the asymptotic formulas for Riccati-Bessel functions

\[ \xi_\nu(p) \sim (-i)^{n+1} \exp(ip), \quad \xi_\nu(p) \sim i^{n+1} \exp(-ip), \]

\[ \psi_\nu(p) \sim [\xi_\nu(p) + \zeta_\nu(p)]/2. \]

Substituting Eq. (41) into Eqs. (39) and (40), after some algebra, the expressions for torque can be considerably simplified for lossless medium. They are

\[ \Gamma_i = \text{Re} [\mathcal{N}_i], \quad \Gamma_i = \text{Im} [\mathcal{N}_i], \quad \Gamma_i = \text{Re} [\mathcal{N}_i], \]

where

\[ \mathcal{N}_1 = \frac{2 \pi \epsilon_0 E_0^2}{k_0^3} \sum_{n,m} \rho_{mn} (\bar{a}_{mn} \bar{a}_{m+1,n}^* + \bar{b}_{mn} \bar{b}_{m+1,n}^* - \bar{b}_{mn} \bar{b}_{m+1,n} - \bar{q}_{mn} \bar{q}_{m+1,n}), \]

\[ \mathcal{N}_2 = \frac{2 \pi \epsilon_0 E_0^2}{k_0^3} \sum_{n,m} m (|\bar{a}_{mn}|^2 + |\bar{b}_{mn}|^2 - |\bar{b}_{mn}|^2 - |\bar{q}_{mn}|^2), \]

with $\rho_{mn} = [(n-m)(n+m+1)]^{1/2}$, whereas $\bar{a}_{mn}$, $\bar{b}_{mn}$, $\bar{p}_{mn}$, and $\bar{q}_{mn}$ are given by

\[ \bar{a}_{mn} = a_{mn} - \frac{1}{2} P_{mn}, \quad \bar{p}_{mn} = \frac{1}{2} P_{mn}, \]

\[ \bar{b}_{mn} = b_{mn} - \frac{1}{2} q_{mn}, \quad \bar{q}_{mn} = \frac{1}{2} q_{mn}, \]

with $a_{mn}$, $b_{mn}$, $p_{mn}$, and $q_{mn}$ being coefficients in Eq. (31). We note that similar expressions for lossless background medium were presented by several authors [27–30].
III. NUMERICAL RESULTS

We are now ready to present systematic numerical results. Except otherwise explicitly specified, the results shown are mostly for the case with \( \epsilon_r = \epsilon_i = 2.56 \), roughly a typical value in most experiments. The material birefringence is thus due to the anisotropic parameter \( u \). In our numerical calculation, the EA of the birefringent particle is set to be the \( z \) axis, without loss of generality. The axial symmetry of the permittivity (5) implies that the scattering is independent of the azimuthal angle \( \phi_0 \) of the incident wave vector \( \mathbf{k}_0 \). As a result, for simplicity, we set \( \phi_0 = 0 \), which means \( \phi_0 = \epsilon_y \) and \( \mathbf{k}_0 \) lies in the \( x-y \) plane. The radiation torque is a function of incident angle \( \theta_k \). In addition, as most experiments applied a laser beam with fixed incident wavelength while the size of the particle is changed (see, e.g., Ref. [16]), in our calculation the optical torque is presented in units of \( 2I_0\lambda^3/c \), with \( c \) the light velocity, \( \lambda \) the incident wavelength, and \( I_0 = \frac{1}{2} c |E_0|^2 \) the incident irradiance.

A. Oblique illumination

We first study the case of oblique illumination, namely, the case with \( \mathbf{k}_0 \) not normal to the EA of the birefringent particle. Three different polarization modes of the incident plane wave are considered: (i) the TM mode, in which \( \hat{\mathbf{p}} = \hat{\mathbf{y}} \) and the electric vector vibrates in the incident plane; (ii) the transverse electric (TE) mode, in which \( \hat{\mathbf{p}} = \hat{\mathbf{y}} \) and thus the magnetic vector vibrates in the incident plane; and (iii) the left circularly polarized (LCP) mode with \( \hat{\mathbf{p}} = (1/\sqrt{2})(\hat{\mathbf{e}}_y + i\hat{\mathbf{e}}_x) \). The incident plane is defined by the EA and \( \mathbf{k}_0 \). We do not study the right circularly polarized incident mode, since its results can be easily inferred from those for the LCP mode by symmetry.

The typical dependence of the optical torque on the incident angle \( \theta_k \) is shown, for \( 0 \leq \theta_k \leq \pi/2 \), in Fig. 2 at size parameter \( \eta = k_d a = 5 \) with different values of birefringent parameter \( u \). For \( \pi/2 \leq \theta_k \leq \pi \), one has \( \Gamma_y(\theta_k) = -\Gamma_y(\pi - \theta_k) \) and \( \Gamma_x(\theta_k) = \Gamma_x(\pi - \theta_k) \) by symmetry. For TM and LCP incident modes, the \( y \) component of the optical torque \( \Gamma_y \) is found to display similar \( \theta_k \) dependence. When \( u > 0 (u < 0) \), corresponding to \( n_r > n_i (n_r < n_i) \), \( \Gamma_y \) is negative (positive), indicating that the optical torque due to the TM incidence will align the EA with the direction parallel (normal) to the \( E \) field, and the CP incidence will make the EA parallel (normal) to the plane of the incident \( E \) field. Stable equilibrium is reached at \( \theta_k = \pi/2 (\theta_k = 0) \) for \( u > 0 (u < 0) \) case. For the TM incident wave, the angle between the EA (z axis) and the incident electric field is \( \varphi_e = (\pi/2) - \theta_k \), as shown in Fig. 2. However, the maximum value of \( \Gamma_y \) does not occur at \( \varphi_e = \theta_k = \pi/4 \), but instead depends on the material birefringence (the anisotropy parameter \( u \)). This is different from the normal illumination case with \( \mathbf{k}_0 \) perpendicular to the EA, where the torque always reaches its maximum at \( \varphi_e = \pi/4 \) (see Fig. 5 below) independent of the value of \( u \). For the TE case, \( \Gamma_y \) is much smaller than in the TM case, and the \( \theta_k \) dependence becomes somewhat complicated, especially for large \( u \), as can be seen in Fig. 2(b). For the case with \( u = 0.4 \), e.g., \( \Gamma_y \) changes sign as \( \theta_k \) goes from 0 to \( \pi/2 \), resulting in two stable equilibrium states at both \( \theta_k = 0 \) and \( \theta_k = \pi/2 \), quite different from the case of \( u = 0.2 \), where only \( \theta_k = 0 \) is rotationally stable. While for the TE and TM cases, \( \Gamma_y \) is the only nonvanishing component of the radiation torque, for the LCP incidence, besides \( \Gamma_y \), the incident wave also exerts an exact \( \Gamma_y \) on the particle, trying to make the EA rotate together with the \( E \) field. When \( \eta \) and \( u \) are not very large, the magnitude of \( \Gamma_y \) monotonically increases with the incident angle \( \theta_k \) in the region from 0 to \( \pi/2 \), and reaches a maximum at \( \theta_k = \pi/2 \); then it decreases with \( \theta_k \) in the region from \( \pi/2 \) to \( \pi \) due to symmetry \( \Gamma_y(\theta_k) = \Gamma_y(\pi - \theta_k) \).

Figure 3 shows \( \Gamma_y/\eta^2 \) and \( \Gamma_y/\eta^2 \) versus \( \theta_k \) at \( u = 0.2 \) for different values of size parameter \( \eta \). Here, \( \Gamma_y/\eta^2 \) and \( \Gamma_y/\eta^2 \), instead of \( \Gamma_y \) and \( \Gamma_y \), versus \( \theta_k \) are exhibited for the convenience of displaying curves for different \( \eta \) in the same figure. For the TM and LCP incident waves, the \( \theta_k \) dependence of \( \Gamma_y \) is similar to the case of small \( \eta \), implying that, irrespective of the value of \( \eta \) for the TM case, the optical torque always tends to make the EA parallel (normal) to the incident \( E \) field for particle with \( n_r > n_i \), and the LCP incidence will align the EA with the directions parallel (normal) to the \( E \)-field plane for particle with \( n_r < n_i \).
FIG. 3. Dependence of $\Gamma_\|^/\eta$ and $\Gamma_\|^/\eta^3$ on $\theta_k$ at $u=0.2$ and different values of $\eta$. (a) $10^3\Gamma_\|^/\eta$ versus $\theta_k$ for the TM incidence; (b) $10^3\Gamma_\|^/\eta^3$ versus $\theta_k$ for the TE incidence; (c) $10^3\Gamma_\|^/\eta^3$ versus $\theta_k$ for the LCP incidence; (d) $10^3\Gamma_\|^/\eta^3$ versus $\theta_k$ for the LCP incidence.

To study the particle size dependence of the optical torque in the small particle limit, we consider also the general LP incidence, in addition to the TM, TE, and LCP modes. Here, the general LP mode is characterized by the polarization vector $\hat{p}=p_p \hat{\theta}_p + p_\phi \hat{\phi}_\phi$, with both $p_p$ and $p_\phi$ real and neither $p_p$ nor $p_\phi$ vanishing. It is taken into account in order to address the particle size dependence of $\Gamma_\|^$, which is vanishing for both the TM ($p_p=0$) and the TE ($p_\phi=0$) cases. Both $\Gamma_\|^$ and $\Gamma_\|^$ versus particle size are found to exhibit different power laws for different incident polarization modes, as exemplified in Fig. 4 for $\theta_k=55^\circ$ and $u=0.2$. Figure 4(a) and Fig. 4(b) display $\Gamma_\|^/\eta^3$ versus $\log_{10}\eta$ for the LCP incidence and the TE case, while Fig. 4(c) and Fig. 4(d) show, respectively, the plots of $\Gamma_\|^/\eta$ and $\Gamma_\|^/\eta^3$ versus $\log_{10}\eta$ for the TE and other polarization modes studied. From Fig. 4(c) and Fig. 4(d), $\Gamma_\|^/\eta^3$ for LP and $\Gamma_\|^/\eta^3$ for LCP incidences. Also shown as dotted line is $10^3\Gamma_\|^/\eta^3$ versus $\log_{10}\eta$ for the LCP case at matching case $\epsilon_r=1$.

FIG. 4. (a) $10^3\Gamma_\|^/\eta$ versus $\log_{10}\eta$ at $u=0.2$ and incident angle $\theta_k=55^\circ$ for the LP incidence with $\hat{p}=(1/\sqrt{2})(\hat{\theta}_k+\hat{\phi}_\phi)$; (b) $10^3\Gamma_\|^/\eta^3$ versus $\log_{10}\eta$ at $u=0.2$ and $\theta_k=55^\circ$ for the LP incidence with $\hat{p}=(1/\sqrt{2})(\hat{\theta}_k+\hat{\phi}_\phi)$; (c) $10^3\Gamma_\|^/\eta^3$ versus $\log_{10}\eta$ at $u=0.2$ and $\theta_k=55^\circ$ for the TE incidence with $\hat{p}=(\hat{\theta}_k+\hat{\phi}_\phi)$; (d) $10^3\Gamma_\|^/\eta^3$ versus $\log_{10}\eta$ at $u=0.2$ and $\theta_k=55^\circ$ for the LP, LCP, and TM incidences. Also shown as dotted line is $10^3\Gamma_\|^/\eta^3$ versus $\log_{10}\eta$ for the LCP case at matching case $\epsilon_r=1$.

B. Normal illumination

Next, we turn to the normal illumination with incident $k_0$ perpendicular to the EA, which has been intensively studied...
FIG. 5. Torque as a function of angle $\varphi_e$ between the extraordinary axis and incident $E$ field for normal illumination and $u=\pm 0.2$ at $\eta=5$ (a) and $\eta=15$ (b). The circles and squares are calculated results, while the lines represent $\Gamma_e=\Gamma_0 \sin 2\varphi_e$. Also shown is the schematic plot for the directions of incident wave vector and $E$ field.

experimentally [12–19]. To be specific, the incident wave is characterized by (22) with $\theta_i=\pi/2$ and $\phi_h=0$, namely, $\mathbf{k}_0$ is in the $x$ direction while keeping the EA in the $z$ direction. For normal illumination, both the TE and the TM incident modes yield vanishing torque. We therefore consider two cases. The first case is the LP mode with the polarization vector given by $\mathbf{p} = \cos \varphi_e \mathbf{\hat{e}}_x + \sin \varphi_e \mathbf{\hat{e}}_y$, so that the angle between the EA and the incident $E$ field is $\varphi_e$, as shown in Fig. 5. The TM and TE modes correspond to $\varphi_e = 0$ and $\varphi_e = \pi/2$, respectively. The second case is the LCP incident wave with the polarization vector $\mathbf{p} = (1/\sqrt{2})(\mathbf{\hat{e}}_x + i \mathbf{\hat{e}}_y)$.

Figure 5 shows the torque as a function of $\varphi_e$ at $\eta=5$ and 15 for the LP incident wave. The circles and squares denote the calculated optical torque, while the lines represent $\Gamma_0 \sin 2\varphi_e$, curves, with $\Gamma_0$ the maximum magnitude of the torque. The sine dependence of torque on $2\varphi_e$ is found to hold up to numerical accuracy, with maximum magnitude of torque occurring at $\varphi_e = \pi/4$, as Eq. (2) proposed in Ref. [18] [see also the first term in (1)]. Our numerical results suggests that the $\Gamma_0 \sin 2\varphi_e$ law is valid for arbitrary size parameter $\eta$ and material birefringence, provided that $\mathbf{k}_0$ is perpendicular to the EA. In addition, it is noted that when $\eta$ is big enough, the EA of particle with $n_r > n_o (u > 0)$ is aligned with the direction normal to the incident $E$ field, in contrast with the case of small $\eta$ experimentally studied [18]. This is clearly seen from Fig. 5. When $u=0.2$, the maximum magnitude of the torque $\Gamma_0 > 0$ for $\eta=5$, implying that the particle will rotate in the positive-$x$ direction. The stable equilibrium is achieved at $\varphi_e = 0$ or $\pi$, resulting in the alignment of the EA with the $E$ field for $n_r > n_o$. For $u=-0.2$, $\Gamma_0 < 0$, a similar analysis leads to the conclusion that the EA is reoriented toward the direction normal to the $E$ field for $n_e < n_o$. The situation becomes contrary for the case with $\eta=15$, as shown in Fig. 5(b), where $\varphi_e=0$ and $\pi$ correspond to the stable equilibrium for $n_r < n_o$. For $n_r > n_o$, $\varphi_e=\pi/2$ is the stable equilibrium, implying that the optical torque tends to make the EA normal to the incident $E$ field.

Figure 6 shows $\Gamma_0$ for the LP incidence as a function of the size parameter $\eta$. It is found that, as the particle size increases, the torque will change its sign, implying that the particle with $n_r > n_o$ is not always aligned with the $E$ field. In addition, $\Gamma_0/\eta$ shows an overall wavelike behavior between positive and negative values, somewhat resembling the first term in (1) based on ray optics for a simple, ideally flat disk. In most cases, $\Gamma_0$ has different signs for $n_r - n_o = \pm |\Delta n|$, indicating that particles with $n_r - n_o = +|\Delta n|$ and $n_r - n_o = -|\Delta n|$ are subject to torque in opposite directions, which is in agreement with the first term in (1). However, it happens at some peculiar ranges of $\eta$ that the torque may have the same sign for both $n_r - n_o = \pm |\Delta n|$, as shown, e.g., in Fig. 6(a) for $|\Delta n|=0.1$ at $\eta \sim 20$ and $\eta \sim 37$. This suggests that the EA may be aligned with the directions parallel or normal to the $E$ field for both $n_r - n_o = \pm |\Delta n|$. The situation presents a striking contrast to the first term in (1), since the latter implies that the particles with $n_r - n_o = \pm |\Delta n|$ are always subject to torque in opposite directions. The difference comes from the effect of particle shape. Finally, it is also noted that the fluctuating behavior in $\Gamma_e$ versus $\eta$ curve is due to nonmatching $\epsilon_e \neq 1$. For the matching case $\epsilon_e = 1$, the wavelike behavior becomes quite smooth, provided that $|\Delta n|$ is not very large, as can be seen in Fig. 6(c).

For the LCP incident wave, Fig. 7 shows plots of $\Gamma_e/\eta^2$ versus $\eta$ for different material birefringences. Qualitative agreement is observed with the second term in (1). The difference lies in that the torque will never vanish except in the limit $\eta \rightarrow 0$, and the amplitude of the wavelike behavior decays with $\eta$ at relative small $\eta$. The overall $\eta$ dependence is similar to the LP case, except that $\Gamma_e$ is always positive instead of fluctuating between positive and negative values in the LP case. This is because the LCP wave always transports positive angular momentum to the scatterer, while the LP wave can transfer both positive and negative angular momentum to the scatterer. In contrast to the LP case shown in
The waveplate behavior is thus verified in the transfer of the angular momentum from the incident light to the scatterer.

To study dependence of the radiation torque on particle size in the small particle limit, Fig. 9 shows the plots of \( \Gamma / \eta^2 \) and \( \Gamma_x / \eta^2 \) versus log\(_{10} \eta \) for, respectively, the LP and LCP incident waves. An incident polarization-dependent scaling law is clearly exhibited: the torque due to the LP incidence obeys the scaling law \( \Gamma_0 \sim \alpha^3 \), as exhibited in Fig. 9(a) for some typical cases with \( \alpha = \pm 0.2 \) and \( \pm 0.4 \), while for the LCP incidence, the torque satisfies \( \Gamma_x \sim \alpha^3 \), as shown typically in Fig. 9(b) for \( \alpha = \pm 0.2 \) and \( \pm 0.4 \).

Figure 10 displays the material birefringence dependence of the optical torque in the case of normal illumination for both the LP and LCP incident modes. The change of the material birefringence is due solely to \( u \) while \( \epsilon_r \) is kept at 2.56. For the LP incidence, we present results for \( \Gamma_x = \epsilon_r \) at \( \varphi = \pi/4 \), namely, \( \Gamma_x \). The \( \Gamma / \eta^2 \) versus \( u \), instead of \( \Gamma \) versus \( u \), is displayed for the convenience of plotting curves for different values of \( \eta \) in the same figure. It is seen that there exists a monotonic increase region at small (absolute) value of \( u \), in which the torque increases with the material birefringence, consistent with the experiment [16]. As \( |u| \) increases, however, different oscillatory behaviors show up for different values of \( \eta \), leading to appearance of many peaks in the range of \( u \) studied. As \( \eta \) increases, the monotonic increase region near \( u=0 \) shrinks, in qualitative agreement with (1). In addition, it is seen that \( \Gamma_x \) is always positive for the LCP incident wave. For the LP case, the torque may change its sign as the material birefringence increases, indicating that the EA of a birefringent particle with \( n_r < n_o \) can also be oriented toward the \( E \)-field direction, even at relative small \( \eta \), provided that the material birefringence is strong enough.

Figure 11 shows the typical torque versus the material birefringence behavior for small [\( \Delta n \)]. The torque due to the LP incident wave is found to display a linear dependence on \( |\Delta n| \), while the torque by the CP incidence exhibits square law behavior \( \Gamma \sim |\Delta n|^2 \), both independent of the particle size. This dependence of \( \Gamma \) on \( \Delta n \) is in unexpected agreement with Eq. (1), although the latter is derived based on the ray optics for a flat disk.

Finally, in Fig. 12 the complex oscillatory behavior of \( \Gamma_0 \) versus \( \epsilon_r = \epsilon_x / \epsilon_0 \) is shown for the LP incidence at \( u = \pm 0.2 \) and different values of \( \eta \). The appearance of sharp peaks recalls the Mie-type resonances. For small \( \eta \), in the range of \( \epsilon_r \) shown, it is seen that the torque does not change its sign. For large \( \eta \), the torque may change its sign as \( \epsilon_r \) increases.
multiplied by −1 and −1

for the LP incidence; to fit all curves for different u in the same figure, results for the LP cases with $u=-0.2$ and $u=-0.4$ have been multiplied by $-1$ and $-\frac{1}{2}$, respectively, while the result for the LCP case with $u=-0.4$ has been multiplied by $\frac{1}{\sqrt{2}}$.

IV. SUMMARY

In summary, we have performed an exact ab initio calculation of the optical torque on a spherical birefringent particle of arbitrary size illuminated by plane electromagnetic wave with arbitrary polarization mode and direction of propagation, based on the extended Mie theory and the Maxwell stress tensor formalism. The expression for evaluating the radiation torque was derived for arbitrary (absorbing and lossless) isotropic surrounding medium. The radiation torque is found to exhibit miscellaneous dependences on the incident angle, the incident polarization, the material birefringence, as well as the particle size. The numerical results confirm the wave-plate mechanism in the transfer of angular momentum from the incident light to the scatterer. When a particle with $n_e > n_o$ ($n_e < n_o$) is subject to oblique illumination by a CP incident wave, its EA will always be aligned with the direction parallel (normal) to the $E$ plane. An obliquely incident TM wave will reorient the EA of the scattering particle to be the $x$ axis different power-law dependences on the particle size in the small particle limit. To be specific, set the EA of the scattering particle to be the $z$ axis and let the incident $k_0$ be in $x-z$ plane. For oblique illumination, $\Gamma_y$ is found to satisfy $\Gamma_y \sim a^6$ and $\Gamma_y \sim a^3$ for the CP and LP incidences, respectively, while $\Gamma_y$ obeys $\Gamma_y \sim a^3$ for all incident polarization modes except the TE case, where $\Gamma_y$ displays $\Gamma_y \sim a^5$ law, besides being much smaller than in other cases. For normal illumination, $\Gamma_y$ vanishes for any polarization mode, $\Gamma_y$ behaves as in the case of oblique illumination, namely, $\Gamma_y \sim a^6$, with the exponent $\alpha=3$ for the CP incidence and $\alpha=6$ for the CP incidence. Finally, when a particle with small material birefringence $\Delta n$ is subject to normal LP and CP illuminations, the optical torque versus $\Delta n$ displays linear and square law behaviors, respectively, regardless of the particle size.

Our direct classical calculation is merited due to subtleties of both classical and quantum theories of electromagnetic angular momentum. Furthermore, it is believed to be relevant to many applications where birefringent spherical particles are implemented, typically in the case of making measurements of viscosity on a microscopic scale [14]. We note that, in reality, the experiments on optical torque are performed using a laser beam. In most cases, however, the re-

FIG. 9. (a) $10^3 \eta_i^3$ versus $\log_{10} \eta$ at different values of $u$ for the LP incidence; (b) $10^3 \eta_i^2$ versus $\log_{10} \eta$ at different values of $u$ for the LCP incidence. To fit all curves for different $u$ in the same figure, results for the LP cases with $u=-0.2$ and $u=-0.4$ have been multiplied by $-1$ and $-\frac{1}{2}$, respectively, while the result for the LCP case with $u=-0.4$ has been multiplied by $\frac{1}{\sqrt{2}}$.

FIG. 10. The optical torque versus $u$ at different values of $\eta$ for the LP (a) and the LCP (b) normal illumination.
results can be understood using the plane-wave picture \[12\], especially when the scattering particles are small compared with the beamwidth and located near the beam axis. In addition, our formulation provides a limiting case against which the solution of more complicated problems (e.g., torque on a particle illuminated by strongly focused laser beam) can be checked. Work along this line is in progress.

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APPENDIX: SOME INTEGRALS INVOLVING \(\pi_{mn}\) AND \(\tau_{mn}\)

Two auxiliary functions, \(\pi_{mn}(\cos \theta)\) and \(\tau_{mn}(\cos \theta)\), are defined by

\[
\pi_{mn}(\cos \theta) = \frac{m}{\sin \theta} P_n^m(\cos \theta),
\]

\[
\tau_{mn}(\cos \theta) = \frac{d}{d\theta} P_n^m(\cos \theta), \quad (A1)
\]

where the first kind of associated Legendre function \(P_n^m(x)\) is given by \[22–24\]

\[
P_n^m(x) = \frac{1}{2^m m!} (1-x^2)^{m/2} \frac{d^{m+1}}{dx^{m+1}}[(x^2-1)^m]. \quad (A2)
\]

The integrals \(\mathcal{W}_{\ell_{\parallel}}^{x,y,z}\) and \(\mathcal{W}_{\ell_{\perp}}^{x,y,z}\) are

\[
\mathcal{W}_{\ell_{\parallel}}^{x,y,z} = \frac{v(u+1)}{|\eta|^2} E_{\ell_{\parallel}}^{\ast} \int \int [i \pi_{mn}(\cos \theta) P_n^m(\cos \theta) \cos \theta \cos \phi + \tau_{mn}(\cos \theta) P_n^m(\cos \theta) \sin \phi] e^{i(n+u)\phi} d\Omega
\]

\[
= i \frac{2\pi}{|\eta|^2} |E_0|^2 \left[ \delta_{\ell_{\parallel},\ell_{\perp}} \delta_{u,u+1} + \left( (n-m)(u+u) \right)^{1/2} \delta_{\ell_{\parallel},\ell_{\perp}} \delta_{u,m+1} + \left( (n+m)(u-u) \right)^{1/2} \delta_{\ell_{\parallel},\ell_{\perp}} \delta_{u,m-1} \right].
\]
\[ \Psi^{(5)}_{np} = \frac{i}{|\eta^2|} E_{mn} E_{\mu\nu} \int \left[ \begin{array}{c} \tau_{mn}(\cos \theta) P^\mu_p(\cos \theta) \cos \phi \tau_{mn}(\cos \theta) P^\mu_p(\cos \theta) \cos \phi \end{array} + \frac{2D}{|\eta^2|} \right] E_0^2 \left[ [(n-m)(v+u)]^{1/2} \delta_{v,u} \delta_{a,1+m} - [(n+m)(v-u)]^{1/2} \delta_{v,u} \delta_{a,1-m} \right] \sin \theta d\Omega = -i \frac{4M \pi}{|\eta^2|} E_0^2 \delta_{v,u} \delta_{a,m} \sin \theta d\Omega. \]

\[ \Psi^{(6)}_{np} = \frac{i}{|\eta^2|} E_{mn} E_{\mu\nu} \int \left[ \begin{array}{c} \tau_{mn}(\cos \theta) P^\mu_p(\cos \theta) \cos \phi - i \tau_{mn}(\cos \theta) P^\mu_p(\cos \theta) \sin \phi \end{array} + \frac{2D}{|\eta^2|} \right] E_0^2 \left[ [(n-m)(v+u)]^{1/2} \delta_{v,u} \delta_{a,1+m} - [(n+m)(v-u)]^{1/2} \delta_{v,u} \delta_{a,1-m} \right] \sin \theta d\Omega = -i \frac{4M \pi}{|\eta^2|} E_0^2 \delta_{v,u} \delta_{a,m} \sin \theta d\Omega. \]

\[ \Psi^{(7)}_{np} = \frac{i}{|\eta^2|} E_{mn} E_{\mu\nu} \int \left[ \begin{array}{c} \tau_{mn}(\cos \theta) P^\mu_p(\cos \theta) \cos \phi + i \tau_{mn}(\cos \theta) P^\mu_p(\cos \theta) \sin \phi \end{array} + \frac{2D}{|\eta^2|} \right] E_0^2 \left[ [(n-m)(v+u)]^{1/2} \delta_{v,u} \delta_{a,1+m} + [(n+m)(v-u)]^{1/2} \delta_{v,u} \delta_{a,1-m} \right] \sin \theta d\Omega = -i \frac{4M \pi}{|\eta^2|} E_0^2 \delta_{v,u} \delta_{a,m} \sin \theta d\Omega. \]


